Announcements

1) Notation on Webwork

$$\frac{\partial(x,y)}{\partial(s,t)} = \text{Jacobian: } \det\begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}$$

2) Exam 3 Solutions online under "files" on Canvas

3) PIC Math - will count for dual engineering or CIS credit

The Definite Integral

Let $B = [a,b] \times [c,d] \times [e,f]$

be a box in $\mathbb{R}^3$. If

$f = f(x,y,z)$ is continuous

on B and real-valued,

define

volume

$$\iint_B f(x,y,z)\, dV \text{ as}$$

$$\int_B f(x,y,z)\,dV$$

$$= \lim_{n\to\infty} \lim_{m\to\infty} \lim_{l\to\infty} \frac{b-a}{n} \frac{d-c}{m} \frac{f-e}{l} \cdot \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} f(x_{i,j,k}, y_{i,j,k}, z_{i,j,k}) \right)$$

where $(x_{i,j,k}, y_{i,j,k}, z_{i,j,k})$

is a point in

$$\left[a+\frac{(j-1)(b-a)}{n}, a+\frac{j(b-a)}{n}\right] \times \left[c+\frac{(i-1)(d-c)}{m}, c+\frac{i(d-c)}{m}\right] \times$$

$$\left[e+\frac{(k-1)(f-e)}{l}, e+\frac{k(f-e)}{l}\right]$$

Note: 1) volume – the volume of the region of integration E can be written as

$$\iiint_E 1 \, dV = \text{vol}(E)$$

2) visualization – the graph of $w = f(x,y,z)$ is 4-dimensional. Very to visualize! You can visualize the region of integration E.

Fubini's Theorem

Suppose $E = [a,b] \times [c,d] \times [e,f]$

and f is continuous on E.

Then

$$\boxed{\begin{aligned}&\iiint_E f(x,y,z)\,dV \\ &= \int_a^b \int_c^d \int_e^f f(x,y,z)\,dz\,dy\,dx \\ &= \int_c^d \int_e^f \int_a^b f(x,y,z)\,dx\,dz\,dy\end{aligned}}$$

plus four other iterations!

Example 1: $f(x,y,z) = x^3 y^2 z^6$

$B = [0,2] \times [-1,1] \times [1,3]$

Compute $\displaystyle\iiint_B x^3 y^2 z^6 \, dV$

Just like in 2-dimensions,

$$\iiint_B x^3 y^2 z^6 \, dV = \int_0^2 x^3 \, dx \cdot \int_{-1}^{1} y^2 \, dy \cdot \int_1^3 z^6 \, dz$$

$$\iiint_B x^3 y^2 z^6 \, dV$$

$$= \int_0^2 x^3 \, dx \cdot \int_{-1}^1 y^2 \, dy \cdot \int_1^3 z^6 \, dz$$

$$= \frac{x^4}{4}\Big|_0^2 \cdot \frac{y^3}{3}\Big|_{-1}^1 \cdot \frac{z^7}{7}\Big|_1^3$$

$$= \boxed{4 \cdot \left(2/3\right) \cdot \left(\frac{3^7}{7} - \frac{1}{7}\right)}$$

In general, if

$$\boxed{f(x,y,z) = h(x)\cdot g(y)\cdot k(z)},$$

then if $B = [a,b]\times[c,d]\times[e,f]$,

$$\boxed{\iiint_B f(x,y,z)\,dV = \int_a^b h(x)\,dx \int_c^d g(y)\,dy \int_e^f k(z)\,dz}$$

More general regions:

Just like in $\mathbb{R}^2$, if E is a bounded region in $\mathbb{R}^3$, there is a box B containing E. To compute

$$\iiint_E f(x,y,z)\,dV,$$ define

$$g(x,y,z) = \begin{cases} f(x,y,z), & (x,y,z) \text{ in } E \\ 0, & (x,y,z) \text{ not in } E \end{cases}$$

Define

$$\boxed{\iint\limits_{E} f(x,y,z)\,dV = \iint\limits_{B} g(x,y,z)\,dV}$$

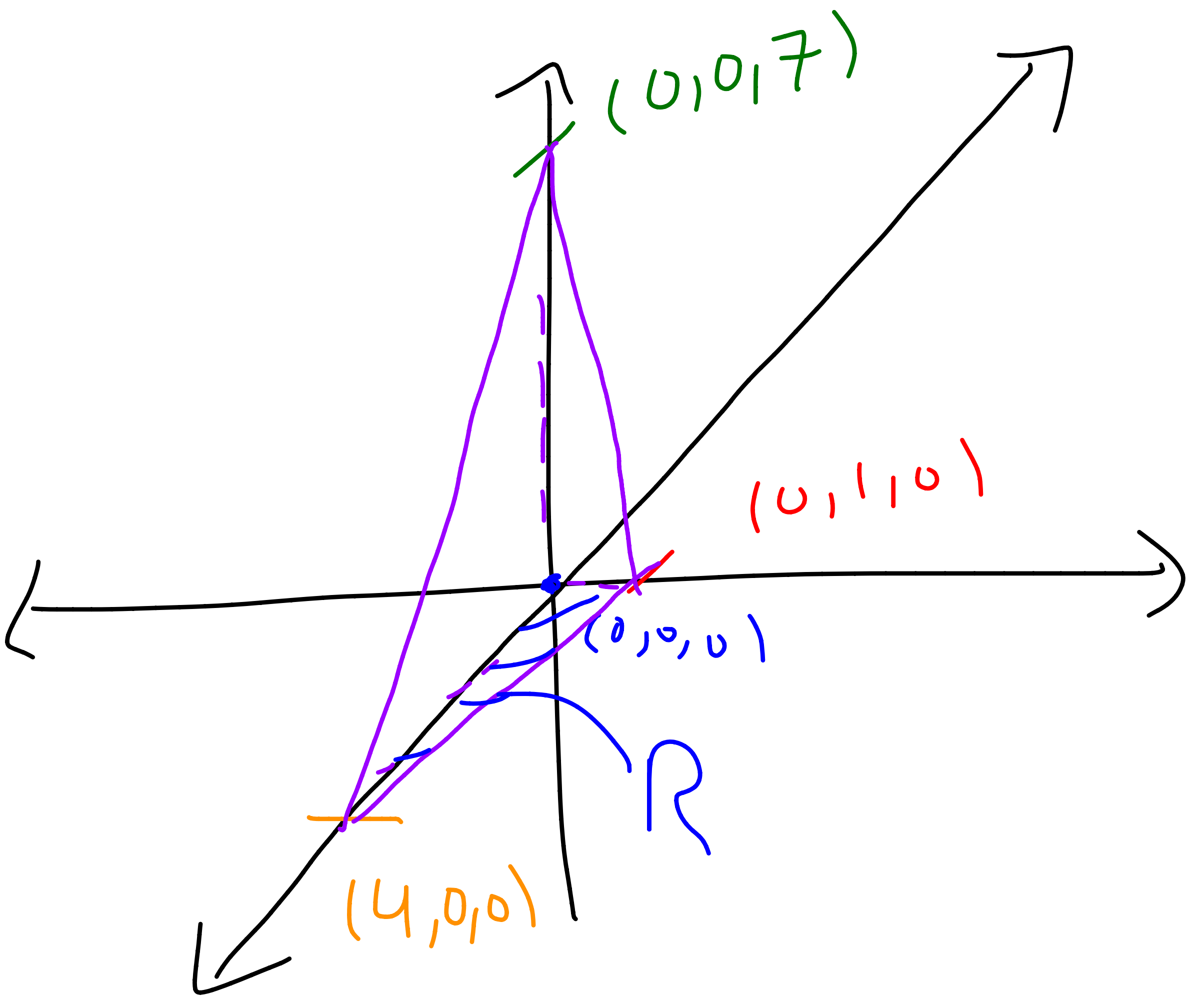
Now we handle integrals over general regions just as for $\mathbb{R}^2$.

Example 2 (tetrahedron)

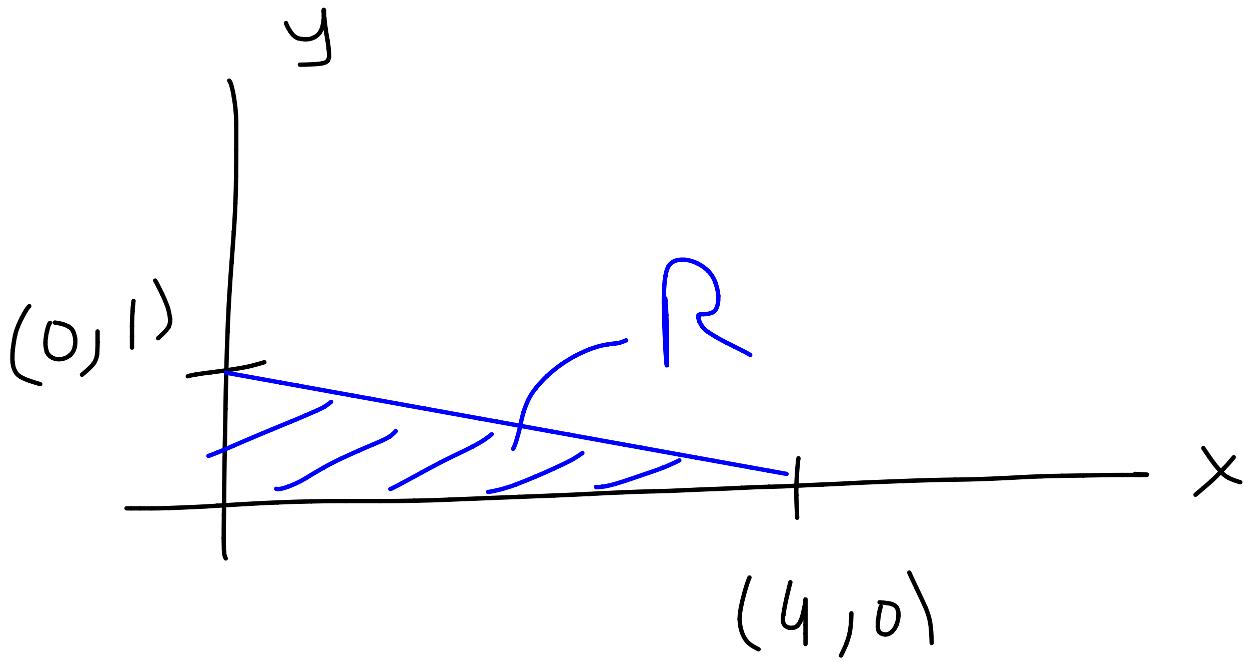
Let E be the tetrahedron with vertices $(0,0,0)$ $(0,0,7)$, $(0,1,0)$, and $(4,0,0)$. Find the volume of E.

$$\mathrm{vol}(E) = \int_E 1 \, dV.$$

Draw E



Describe R in the (x,y)-plane:



$0 \leq x \leq 4$ ← x bounds

$0 \leq y \leq -\frac{x}{4} + 1$ ← y bounds

For the z-bounds,

$0 \leq z \leq$ (plane formed by $(0,0,7)$, $(0,1,0)$, $(4,0,0)$)

Make equation of the plane:

vectors "on" the plane are

$v = (0,0,7) - (4,0,0) = \langle -4, 0, 7 \rangle$

$w = (0,0,7) - (0,1,0) = \langle 0, -1, 7 \rangle$

$\boxed{\text{normal vector} = v \times w}$

$$v \times w = \begin{array}{ccccc} i & j & k & i & j \\ -4 & 0 & 7 & -4 & 0 \\ 0 & -1 & 7 & 0 & -1 \end{array}$$

$$= \langle 7, 28, 4 \rangle$$

Plane:

$$\langle 7, 28, 4 \rangle \cdot \langle x, y-1, z \rangle = 0$$

$$7x + 28y - 28 + 4z = 0, \text{ so}$$

$$\boxed{z = \frac{7x + 28y - 28}{-4}}$$

So

$$0 \leq z \leq \frac{7x+28y-28}{-4} \quad \leftarrow z \text{ bounds}$$

$$= \frac{-7x}{4} - 7y + 7$$

$$\text{vol}(E) = \iiint_E 1 \, dV$$

$$= \int_0^4 \int_0^{-\frac{x}{4}+1} \int_0^{-\frac{7x}{4}-7y+7} 1 \, dz \, dy \, dx$$

$$\mathrm{vol}(E) = \int_E 1 \, dV$$

$$= \int_0^4 \left(\int_0^{-\frac{x}{4}+1} \left(\int_0^{-\frac{7x}{4}-7y+7} 1 \, dz \right) dy \right) dx$$

$$= \int_0^4 \left(\int_0^{-\frac{x}{4}+1} \left(-\frac{7x}{4} - 7y + 7 \right) dy \right) dx$$

$$= \int_0^4 \left(-\frac{7xy}{4} - \frac{7y^2}{2} + 7y \right) \Bigg|_0^{-\frac{x}{4}+1} dx$$

$$\int_0^4 \left(-\frac{7xy}{4} - \frac{7y^2}{2} + 7y\right)\Bigg|_0^{-\frac{x}{4}+1} dx$$

$$= \int_0^4 y\left(-\frac{7x}{4} - \frac{7y}{2} + 7\right)\Bigg|_0^{-\frac{x}{4}+1} dx$$

$$= \int_0^4 \left(-\frac{x}{4}+1\right)\left(-\frac{7x}{4} + \frac{7x}{8} - \frac{7}{2} + 7\right) dx$$

$$= \int_0^4 \left(-\frac{x}{4}+1\right)\left(-\frac{7x}{8} + \frac{7}{2}\right) dx$$

$$= \int_0^4 \left(-\frac{x}{4}+1\right)\left(-\frac{7x}{8}+\frac{7}{2}\right) dx$$

$$= \int_0^4 \left(\frac{7x^2}{32}-\frac{7x}{4}+\frac{7}{2}\right) dx$$

$$= \left(\frac{7x^3}{96}-\frac{7x^2}{8}+\frac{7x}{2}\right)\Bigg|_0^4$$

$$= \frac{448}{96}-\frac{112}{8}+\frac{28}{2} > 0$$

$$= \frac{448}{96} \cancel{- \frac{28}{2} + \frac{28}{2}} = \boxed{\frac{448}{96}}$$

Type Regions in $\mathbb{R}^3$

Let R be a type I or II region in $\mathbb{R}^2$.

Example 3: Compute

$$\iiint_E \sqrt{x^2+z^2}\, dV$$

where E is the region bounded by the paraboloid $y = x^2+z^2$ and the plane $y = 4$.

Transformations on $\mathbb{R}^3$

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$

$$T(x,y,z) = \big(g(x,y,z), h(x,y,z), k(x,y,z)\big)$$

where

$$g, h, k : \mathbb{R}^3 \to \mathbb{R}.$$

The Jacobian

$$J_T(x,y,z)$$

$$= \det\left(\begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial h}{\partial x} & \frac{\partial k}{\partial x} \\ \frac{\partial g}{\partial y} & \frac{\partial h}{\partial y} & \frac{\partial k}{\partial y} \\ \frac{\partial g}{\partial z} & \frac{\partial h}{\partial z} & \frac{\partial k}{\partial z} \end{bmatrix}\right)$$

Change of Variables

If $T: \mathbb{R}^3 \to \mathbb{R}^3$,

E a region in $\mathbb{R}^3$.

Then

$$\int_{T(E)} f(x,y,z)\,dV$$

$$\boxed{= \int_E f(T(x,y,z))\,|J_T(x,y,z)|\,dV}$$

Provided: f is continuous, E is bounded, all the first-order partials of $g, h,$ and k are continuous, and T is one-to-one on E.

Special Coordinate Systems

(Sections 15.8 and 15.9)

Cylindrical Coordinates

Polar coordinates, with a z!

$x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$

$r > 0$, $0 \le \theta < 2\pi$

If

$$T(r,\theta,z) = (r\cos(\theta), r\sin(\theta), z),$$

$$J_T(x,y,z) = r$$

Don't forget the r!

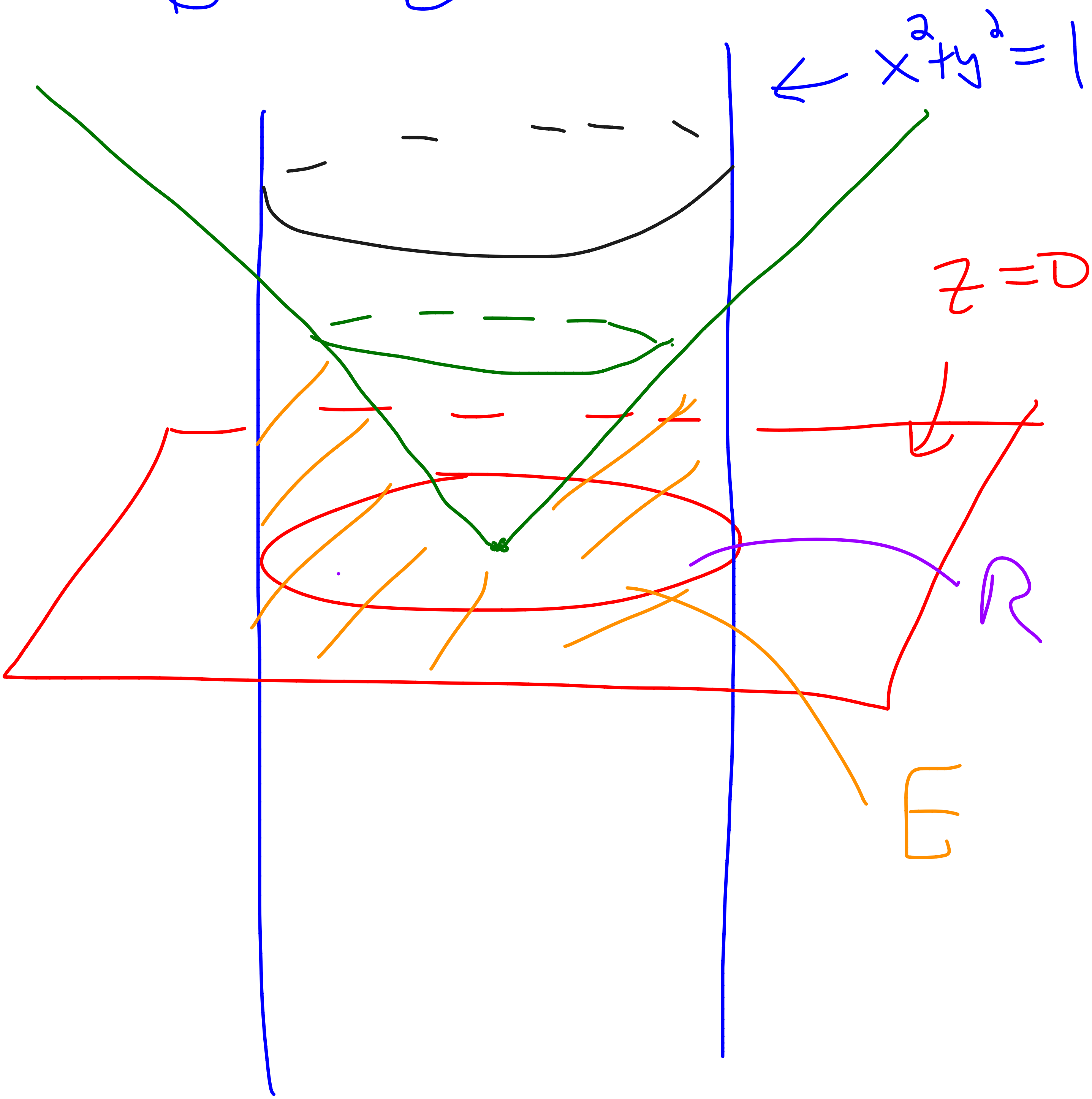
Example 3: $\iiint_E x^2 \, dV$ where

E is the region inside $x^2+y^2=1$, above $z=0$, and below the cone $z^2=4x^2+4y^2$.

Switch to cylindrical coordinates,

first draw E.

Draw E



E = below green, above red, inside blue.

R = region inside $x^2+y^2=1$

= inside $r^2=1$

= inside $r=1$, so

$$\boxed{0 < r \leq 1}$$

You get the whole circle, so

$$\boxed{0 \leq \theta < 2\pi}$$

z values go from xy-plane to the cone $z^2 = 4x^2 + 4y^2$, but only positive z-values occur, so this is

$$\boxed{\begin{aligned} 0 \le z &\le 2\sqrt{x^2+y^2} \\ &= 2r \end{aligned}}$$

In cylindrical coordinates,

$$\iiint_E x^2 \, dV \qquad (x = r\cos(\theta))$$

don't forget!

$$= \int_0^{2\pi} \int_0^{1} \int_0^{2r} r^2 \cos^2\theta \, r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \left(\int_0^{1} \left(\int_0^{2r} r^3 \cos^2\theta \, dz \right) dr \right) d\theta$$

$$= \int_0^{2\pi} \left(\int_0^1 \left(\int_0^{2r} r^3 \cos^2\theta \, dz \right) dr \right) d\theta$$

$$= \int_0^{2\pi} \left(\int_0^1 2r^4 \cos^2\theta \, dr \right) d\theta$$

$$= \int_0^1 2r^4 \, dr \cdot \underbrace{\int_0^{2\pi} \cos^2\theta \, d\theta}$$

$$\int_0^{2\pi} \frac{1+\cos(2\theta)}{2} \, d\theta$$

$$\int_0^{2\pi} \frac{1+\cos(2\theta)}{2}\,d\theta$$

$$= \left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right)\Big|_0^{2\pi}$$

$$= \pi$$

$$\int_0^1 2r^4\,dr = \frac{2r^5}{5}\Big|_0^1 = \frac{2}{5}$$

Final answer: $\boxed{\frac{2\pi}{5}}$

Spherical Coordinates

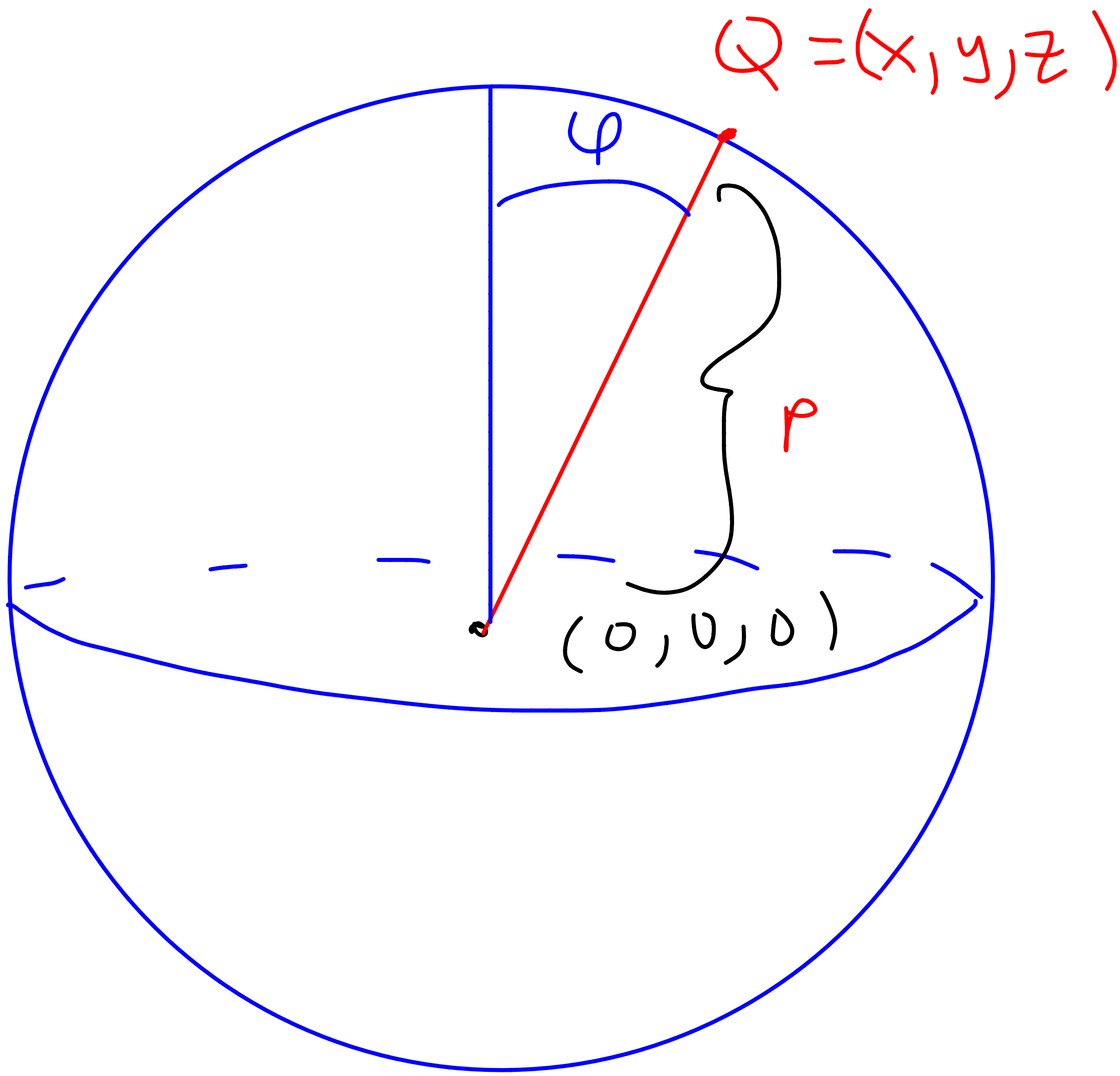
True analog of polar coordinates in $\mathbb{R}^3$:

Describe every point via distance to the

$\rho \rightarrow$ origin, $\underbrace{\text{angle on the xy-plane}}_{\theta}$,

$\underbrace{\text{angle on the yz-plane}}_{\varphi}$

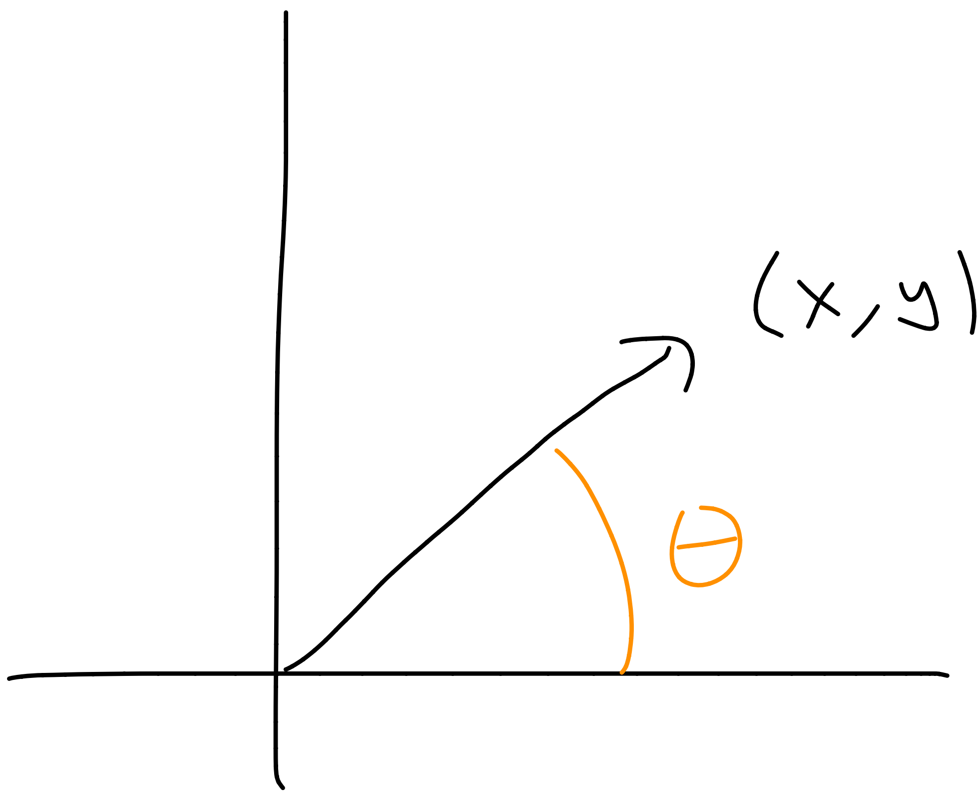
Picture



φ = "up-down" angle

θ = "left-right" angle

To see θ, plot the vector (x,y)



Same θ as in cylindrical coordinates.

Transformation

$$x = \rho\cos\theta\sin\varphi$$

$$y = \rho\sin\theta\sin\varphi$$

$$z = \rho\cos\varphi$$

$$\rho > 0, \quad 0 \le \theta < 2\pi,$$

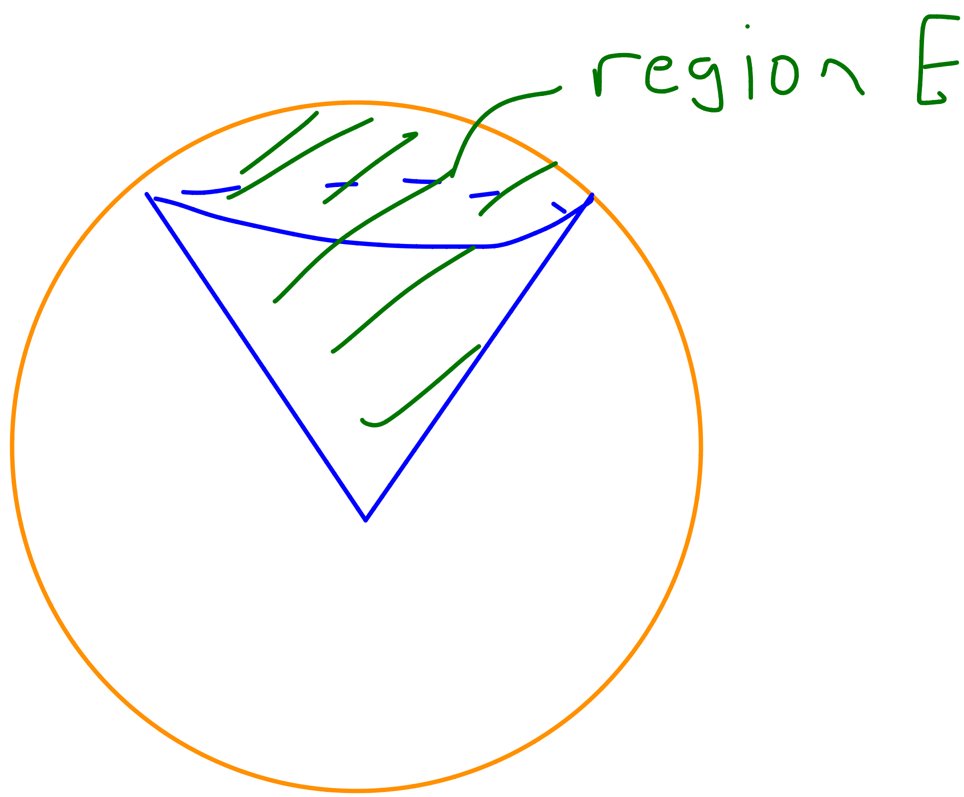
$$0 \le \varphi \le \pi$$

Jacobian: $\rho^2\sin\varphi$ don't forget

Example 4: (#25, section 15.4, revisited!)

Find the volume above $z = \sqrt{x^2+y^2}$ and below (inside) $x^2+y^2+z^2=1$

Picture:



$$\mathrm{vol}(E) = \int_E 1 \, dV$$

Describe E!

Use spherical coordinates

Completely around cone,

so $0 \leq \theta < 2\pi$.

Sphere $x^2+y^2+z^2=1$

becomes $\rho^2=1$, so $\rho=1$,

$0 < \rho \leq 1$.

Finally, the cone

$z = \sqrt{x^2+y^2}$,

$x = \rho \cos\theta \sin\varphi$

$y = \rho \sin\theta \sin\varphi$, so

$x^2+y^2 = \rho^2 \sin^2\varphi$

$z = \rho \sin\varphi$

$$\cancel{\rho}\sin\varphi = \cancel{\rho}\cos\varphi$$

$\sin\varphi = \cos\varphi$, so $\varphi = \frac{\pi}{4}$

is the equation of the

cone! So

$$\text{vol}(E) = \int_0^{2\pi}\int_0^{1}\int_0^{\pi/4} \rho^2 \sin(\varphi)\, d\varphi\, d\rho\, d\theta$$

$$= 2\pi \int_0^1 \rho^2 d\rho \int_0^{\pi/4} \sin(\varphi)\, d\varphi$$

$$= \frac{2\pi}{3}\left(-\cos(\varphi)\Big|_0^{\pi/4}\right)$$

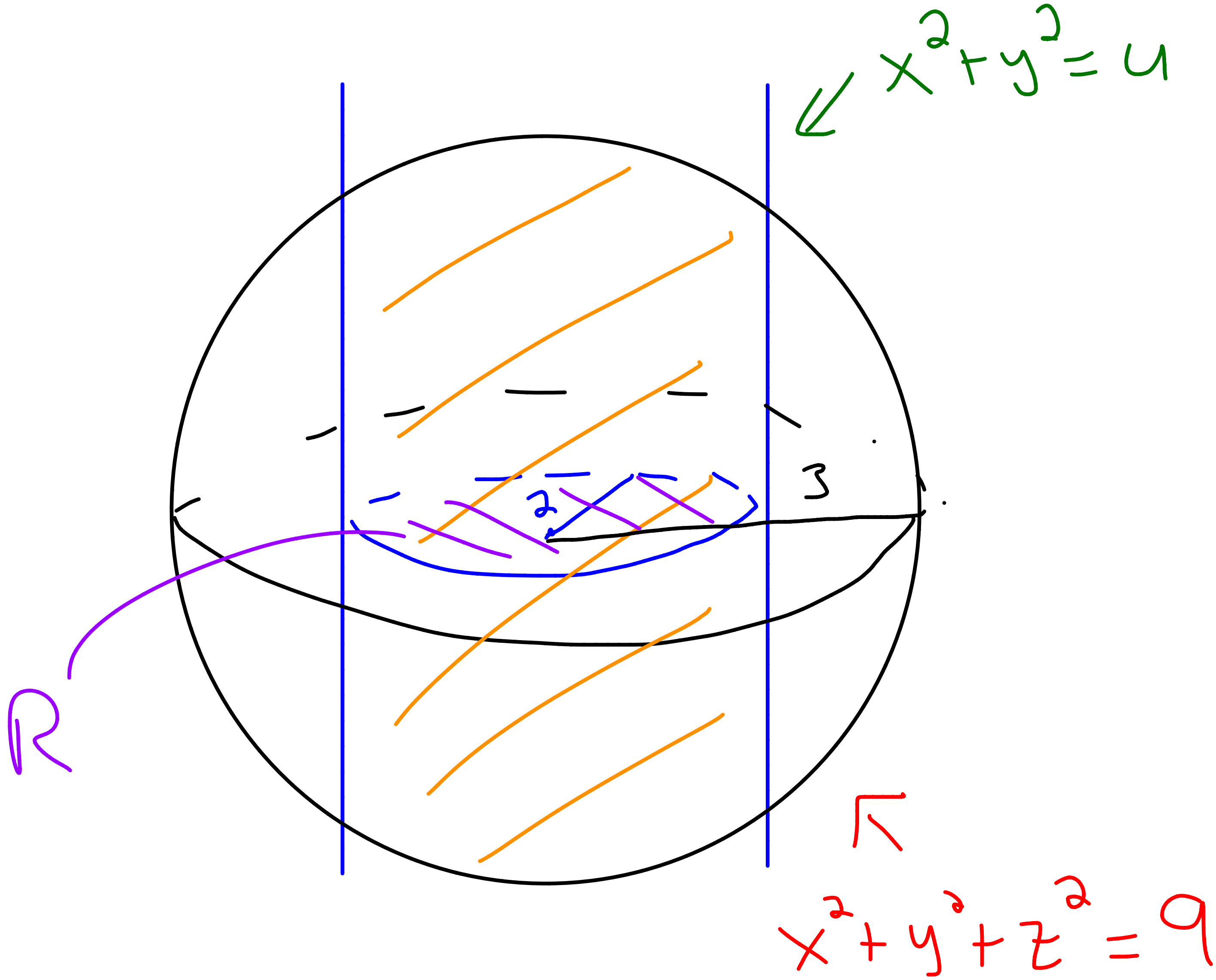
$$= \boxed{\frac{2\pi}{3}\left(1 - \frac{\sqrt{2}}{2}\right)}$$

Example 5: Compute

$$\iiint_E \sin\left(\sqrt{9-x^2-y^2}\right) dV$$

where E is the region inside both $x^2+y^2=4$ and $x^2+y^2+z^2=9$.

Picture



We have both a sphere and a cylinder! Which coordinate system do you use?

Try cylindrical

Region of integration R:

$$x^2+y^2 \leq 4,$$

$$r^2 \leq 4,$$

$$\boxed{\begin{aligned} &0 < r < 2, \\ &0 \leq \theta < 2\pi \end{aligned}}$$

Upper bound on z = top half of sphere,

$$z = \sqrt{9 - x^2 - y^2}$$

$$= \sqrt{9 - r^2}$$

Lower bound = bottom half of sphere $z = -\sqrt{9 - r^2}$

$$\boxed{-\sqrt{9 - r^2} \leq z \leq \sqrt{9 - r^2}}$$

$$\iiint_E \sin\left(\sqrt{9-x^2-y^2}\right) dV$$

$$= \int_0^{2\pi}\left(\int_0^{2}\left(\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} \sin\left(\sqrt{9-r^2}\right) r \, dz\right) dr\right) d\theta$$

$$= \int_0^{2\pi}\left(\int_0^{2} \sqrt{9-r^2}\,\sin\left(\sqrt{9-r^2}\right) 2r \, dr\right) d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{2} \sqrt{9-r^2}\,\sin\left(\sqrt{9-r^2}\right) 2r\,dr$$

$$\int_0^{2\pi} d\theta \int_0^2 \sqrt{9-r^2}\,\sin\left(\sqrt{9-r^2}\right) 2r\,dr$$

$$= 2\pi \int_0^2 \sqrt{9-r^2}\,\sin\left(\sqrt{9-r^2}\right) 2r\,dr$$

Let $u = 9 - r^2$

$u = -2r\,dr$

$u(0) = 9, \; u(2) = 5$

$$= -2\pi \int_9^5 \sqrt{u}\,\sin(\sqrt{u})\,du$$

$$= 2\pi \int_5^9 \sqrt{u}\,\sin(\sqrt{u})\,du$$

For $2\pi \int_5^9 \sqrt{u} \sin(\sqrt{u})\, du$,

let $w = \sqrt{u}$

$$dw = \frac{1}{2\sqrt{u}}\, du$$

$$2\sqrt{u}\, dw = du$$

$$2w\, dw = du$$

$$w(5) = \sqrt{5}, \quad w(9) = 3$$

The integral becomes

$$4\pi \int_{\sqrt{5}}^{3} w^2 \sin(w)\, dw$$

For $4\pi \int_{\sqrt{5}}^{3} w^2 \sin(w)\, dw$,

integrate by parts (tabular trick)

u		dv
w^2	$+$	$\sin(w)$
$2w$	$-$	$-\cos(w)$
2	$+$	$-\sin(w)$
0		$\cos(w)$

we get

$$4\pi\left(-w^2\cos(w) + 2w\sin(w) + 2\cos(w)\right)\Big|_{\sqrt{5}}^{3}$$

$$4\pi\left(-w^2\cos(w)+2w\sin(w)+2\cos(w)\right)\Big|_{\sqrt{5}}^{3}$$

$$= 4\pi\big(-9\cos(3)+6\sin(3)+\cos(3)$$
$$+5\cos(\sqrt{5})-2\sqrt{5}\sin(\sqrt{5})$$
$$-2\cos(\sqrt{5})\big)$$

$$\boxed{=4\pi\left(-8\cos(3)+6\sin(3)+3\cos(\sqrt{5})-2\sqrt{5}\sin(\sqrt{5})\right)}$$